

# Partitions of graphs into small and large sets

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## Abstract

Let  $G$  be a graph on  $n$  vertices. We call a subset  $A$  of the vertex set  $V(G)$  *k-small* if, for every vertex  $v \in A$ ,  $\deg(v) \leq n - |A| + k$ . A subset  $B \subseteq V(G)$  is called *k-large* if, for every vertex  $u \in B$ ,  $\deg(u) \geq |B| - k - 1$ . Moreover, we denote by  $\varphi_k(G)$  the minimum integer  $t$  such that there is a partition of  $V(G)$  into  $t$  *k-small* sets, and by  $\Omega_k(G)$  the minimum integer  $t$  such that there is a partition of  $V(G)$  into  $t$  *k-large* sets. In this paper, we will show tight connections between *k-small* sets, respectively *k-large* sets, and the *k*-independence number, the clique number and the chromatic number of a graph. We shall develop greedy algorithms to compute in linear time both  $\varphi_k(G)$  and  $\Omega_k(G)$  and prove various sharp inequalities concerning these parameters, which we will use to obtain refinements of the Caro-Wei Theorem, the Turán Theorem and the Hansen-Zheng Theorem among other things.

**Keywords:** *k*-small set, *k*-large set, *k*-independence, clique number, chromatic number

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## 1 Introduction and Notation

We start with the following basic definitions. Let  $n$  and  $m$  be two positive integers. Let  $S = \{0 \leq a_1 \leq a_2 \leq \dots \leq a_m \leq n - 1\}$  be a sequence of  $m$  integers and  $\overline{S} = \{0 \leq b_m \leq b_{m-1} \leq \dots \leq b_1 \leq n - 1\}$  be the complement sequence, where  $b_i = n - a_i - 1$  for  $1 \leq i \leq m$ . Let  $k \geq 0$  be an integer. A subsequence  $A$  of  $S$  is called *k-small* if, for every member  $x$

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of  $A$ ,  $x \leq n - |A| + k$ . A subsequence  $B$  of  $S$  is called *k-large* if, for every member  $x$  of  $B$ ,  $x \geq |B| - k - 1$ . In particular, for the terminology of graphs, we have the following definitions. Let  $G$  be a graph on  $n$  vertices. We call a set of vertices  $A \subseteq V(G)$  *k-small* if, for every vertex  $v \in A$ ,  $\deg(v) \leq n - |A| + k$ . A subset  $B \subseteq V(G)$  is called *k-large* if, for every vertex  $v \in B$ ,  $\deg(v) \geq |B| - k - 1$ . When  $k = 0$ , we say that  $A$  is a *small* set ( $\delta$ -set in [13, 1]) and  $B$  a *large* set. Let  $S_k(G)$  denote the maximum cardinality of a *k-small* set and  $L_k(G)$  denote the maximum cardinality of a *k-large* set in  $G$ . Further, given a graph  $G$ , let  $\varphi_k(G)$  be the minimum integer  $t$  such that there is a partition of  $V(G)$  into  $t$  *k-small* sets, and let  $\Omega_k(G)$  be the minimum integer  $t$  such that there is a partition of  $V(G)$  into  $t$  *k-large* sets. When  $k = 0$ , we will set  $\varphi(G)$  instead of  $\varphi_0(G)$  and  $\Omega(G)$  instead of  $\Omega_0(G)$ . Consider the following observations.

**Observation 1.1.** *Let  $n$  and  $m$  be two positive integers. Let  $S = \{0 \leq a_1 \leq a_2 \leq \dots \leq a_m \leq n - 1\}$  be a sequence and let  $G$  be a graph.*

- (i)  *$A$  is a  $k$ -small subsequence of  $S$  if and only if  $\overline{A}$  is a  $k$ -large subsequence of  $\overline{S}$ ;*
- (ii)  *$A$  is a  $k$ -small set in  $G$  if and only if  $A$  is a  $k$ -large set in  $\overline{G}$ ;*
- (iii)  *$S_k(G) = L_k(\overline{G})$  and  $L_k(G) = S_k(\overline{G})$ ;*
- (iv)  *$\varphi_k(G) = \Omega_k(\overline{G})$  and  $\Omega_k(G) = \varphi_k(\overline{G})$ .*

*Proof.* (i)  $A$  is a small subsequence of  $S$  if and only if  $a_i \leq n - |A| + k$  for every  $a_i \in A$ , which is equivalent to  $b_i \geq |A| - k - 1$  for each  $b_i = n - a_i - 1 \in \overline{A}$ , meaning that  $\overline{A}$  is a  $k$ -large subsequence of  $\overline{S}$ .

(ii)  $A$  is a  $k$  small set of  $G$  if and only if  $\deg_G(v) \leq n - |A| + k$  for every  $v \in A$ , which is equivalent to  $\deg_{\overline{G}}(v) = n - 1 - \deg_G(v) \geq |A| - k - 1$  for every  $v \in A$ , meaning that  $A$  is a  $k$ -large set in  $\overline{G}$ .

(iii) and (iv) follow directly from (ii).  $\square$

A *k-independent set*  $A$  in  $G$  is a subset of vertices of  $G$  such that  $|N(v) \cap A| \leq k$  for every  $v \in A$ . The maximum cardinality of a  $k$ -independent set is denoted by  $\alpha_k(G)$ . Note that a 0-independent set is precisely an independent set, so we will use the usual notation  $\alpha(G)$  for the independence number instead of  $\alpha_0(G)$ . The well-known Caro-Wei bound [2, 17]  $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$  was generalized by Favaron [8] to  $\alpha_k(G) \geq \sum_{v \in V(G)} \frac{k}{1+k\deg(v)}$ . Other generalizations and improvements were given in [3, 12]. For more information on the  $k$ -independence number see also the survey [4].

Similarly, we call a subset  $B \subseteq V(G)$  such that  $|N(v) \cap B| \geq |B| - k - 1$  for every  $v \in B$  a *k-near clique* and the cardinality of a maximum  $k$ -near clique will be denoted by  $\omega_k(G)$ . A 0-near clique is precisely a clique and so we will use the usual notation for the clique number  $\omega(G)$  instead of  $\omega_0(G)$ .

The connection between  $k$ -independent sets and  $k$ -near cliques to  $k$ -small and  $k$ -large sets is given below.

**Observation 1.2.** *In a graph  $G$ , every  $k$ -independent set is a  $k$ -small set and every  $k$ -near clique is a  $k$ -large set;*

*Proof.* Let  $A$  be a  $k$ -independent set and  $B$  a  $k$ -near clique of  $G$ . Then  $\deg(v) \leq n - |A| + k$  for every  $v \in A$  and  $\deg(v) \geq |B| - k - 1$  for every  $v \in B$ . Hence,  $A$  is a  $k$ -small set and  $B$  a  $k$ -large set.  $\square$

We denote by  $\deg(v) = \deg_G(v)$  the *degree* of the vertex  $v$  in  $G$  and  $N_G(v)$  and  $N_G[v]$  is its *open* and, respectively, *closed neighborhood* of  $v$ . With  $d(G)$  we refer to the *average degree*  $\frac{1}{|V(G)|} \sum_{v \in V(G)} \deg(v)$  of  $G$ . Given the degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$  of  $G$ , we will denote by  $v_1, v_2, \dots, v_n$  the vertices of  $G$  ordered accordingly to the degree sequence, i.e. such that  $\deg(v_i) = d_i$ . Moreover,  $\chi(G)$  is the chromatic number and  $\theta(G)$  the clique-partition number of  $G$ . For notation not mentioned here, we refer the reader to [18].

The paper is organized in several sections as follows:

- 1 Introduction and Notation
- 2 Bounds on  $S_k(G)$  and  $L_k(G)$  with applications to upper and lower bounds on  $\alpha_k(G)$  and  $\omega_k(G)$
- 3 Algorithms for  $\varphi_k(G)$  and  $\Omega_k(G)$
- 4 Bounds on  $\varphi_k(G)$  and  $\Omega_k(G)$
- 5 More applications to  $\alpha(G)$  and  $\omega(G)$
- 6 Variations of small and large sets
- 7 References

## 2 Bounds on $S_k(G)$ and $L_k(G)$ with applications to upper bounds on $\alpha_k(G)$ and $\omega_k(G)$

Since every  $k$ -independent set of  $G$  is a  $k$ -small set and every  $k$ -near clique of  $G$  is a  $k$ -large set, one expects that the bounds on  $S_k(G)$ ,  $L_k(G)$ ,  $\varphi(G)$  and  $\Omega(G)$  can be derived using their arithmetic definitions, and that some properties will be also useful in obtaining bounds on the much harder to compute  $\alpha_k(G)$  and  $\omega_k(G)$ . As we shall see in the sequel this is indeed the case and several refinements of the Caro-Wei Theorem [2, 17], the Turán Theorem [16] and Hansen-Zheng Theorem [10] are easily derived from bounds using  $k$ -small sets and  $k$ -large sets as well as some relations between  $L_0(G)$  and  $\chi(G)$ . A lower bound on  $\alpha(G)$  and  $\omega(G)$  in terms of  $\Omega(G)$  and  $\varphi(G)$ , respectively, illustrates the usefulness of working with small and large sets.

**Theorem 2.1.** *Let  $G$  be a graph. Then  $\alpha(G) \geq \Omega(G)$  and  $\omega(G) \geq \varphi(G)$ .*

*Proof.* Let  $G_1 = G$  and let  $x_1$  be a vertex of minimum degree in  $G_1$ . Now, for  $i \geq 1$ , let  $x_i$  be a vertex of minimum degree in  $G_i$  and define successively  $G_{i+1} = G_i - N_{G_i}[x_i]$  and  $V_i = N_{G_i}[x_i]$ , until there are no vertices left, say until index  $q$ . In this way, we obtain a partition  $V_1 \cup V_2 \cup \dots \cup V_q$  of  $V(G)$  into large sets, as, for every  $v \in V_i$ ,  $\deg_G(v) \geq \deg_{G_i}(v) \geq \deg_{G_i}(x_i) = |V_i| - 1$ . Hence,  $q \geq \Omega(G)$ . On the other side,  $\{x_1, x_2, \dots, x_q\}$  is an independent set by construction and thus  $\alpha(G) \geq q$ . Therefore,  $\alpha(G) \geq \Omega(G)$  and also  $\omega(G) = \alpha(\overline{G}) \geq \Omega(\overline{G}) = \varphi(G)$  and we are done.  $\square$

We mention that a more complicated proof of  $\omega(G) \geq \varphi(G)$  was given in [13]. One of the strongest lower bounds for the independence number of a graph is the so called residue of the degree sequence denoted  $R(G)$  (see [9, 15, 12]), which is the number of zeros left in the end of the Havel-Hakimi algorithm. As we shall see later, computing  $\Omega(G)$  requires  $O(|V(G)|)$ -time while the Havel-Hakimi algorithm requires  $O(|E(G)|)$ -time. While  $R(G)$  does better than all of the lower bounds given in the survey [19], here are two examples showing that in one case  $R(G)$  does better and in the other  $\Omega(G)$  does better. For the star  $G = K_{1,n}$ ,  $R(G) = n - 1$  while  $\Omega(G) \sim \frac{n}{2}$ . However, for the graph  $G$  on 6 vertices with degree sequence 1, 2, 2, 3, 3, 3,  $\Omega(G) = 3$  while  $R(G) = 2$ .

While the above theorem gives lower bounds on  $\alpha(G)$  and  $\omega(G)$  in terms of  $\Omega(G)$  and  $\varphi(G)$ , the next one gives upper bounds on  $\alpha_k(G)$  and  $\omega_k(G)$  in terms of  $S_k(G)$  and  $L_k(G)$ .

**Theorem 2.2.** *Let  $G$  be a graph on  $n$  vertices and let  $d_1 \leq d_2 \leq \dots \leq d_n$  its degree sequence. Then*

- (i)  $S_k(G) \geq \alpha_k(G)$  and  $L_k(G) \geq \omega_k(G)$ ;
- (ii)  $S_k(G) \geq \frac{n}{\varphi_k(G)}$  and  $L_k(G) \geq \frac{n}{\Omega_k(G)}$ ;
- (iii)  $S_k(G) = \max\{s : d_s \leq n - s + k\}$  and  $\{v_1, v_2, \dots, v_{S_k(G)}\}$  is a maximum  $k$ -small set of  $G$ ;
- (iv)  $L_k(G) = \max\{t : t - k - 1 \leq d_{n-t+1}\}$  and  $\{v_{n-L_k+1}, v_{n-L_k+2}, \dots, v_n\}$  is a maximum  $k$ -large set of  $G$ .

*Proof.* (i) Since a  $k$ -independent set is a  $k$ -small set and a  $k$ -near-clique is a  $k$ -large set,  $S_k(G) \geq \alpha_k(G)$  and  $L_k(G) \geq \omega_k(G)$ .

(ii) Let  $V_1 \cup V_2 \cup \dots \cup V_t$  be a partition of  $V(G)$  into  $t = \varphi_k(G)$   $k$ -small sets. Then  $S_k(G) \geq \max_{1 \leq i \leq t} |V_i| \geq \frac{n}{t} = \frac{n}{\varphi_k(G)}$ . The other inequality follows from  $L_k(G) = S_k(\overline{G}) \geq \frac{n}{\varphi_k(\overline{G})} = \frac{n}{\Omega_k(G)}$ .

(iii) Let  $v_1, v_2, \dots, v_n$  be the vertices of  $G$  ordered according to its degree sequence. Let  $A$  be an arbitrary  $k$ -small set. Clearly, for every vertex  $v \in A$ ,  $\deg(v) \leq n - |A| + k$ . Now order the degrees of the vertices of  $A$  in increasing order such that  $\deg(u_1) \leq \deg(u_2) \leq \dots \leq \deg(u_{|A|}) \leq n - |A| + k$ . Then  $d_{|A|} \leq \deg(u_{|A|}) \leq n - |A| + k$ . Hence, for every  $k$ -small set  $A$ ,  $d_{|A|} \leq \deg(u_{|A|}) \leq n - |A| + k$ . Now let  $s$  be the largest index in the degree sequence

of  $G$  such that  $d_s \leq n - s + k$ . Then  $s \geq S_k(G)$ , as this inequality holds for any  $k$ -small set. But observe that  $\{v_1, v_2, \dots, v_s\}$  is  $k$ -small by definition. Hence  $S_k(G) \geq s$  and we conclude that  $s = S_k(G)$ .

(iv) Let  $\bar{d}_1 \leq \bar{d}_2 \leq \dots \leq \bar{d}_n$  be the degree sequence of  $\bar{G}$  given through  $\bar{d}_i = n - 1 - d_{n-i+1}$  for  $1 \leq i \leq n$ . Then, by item (iii) and Observation 1.1(iii) we have  $L_k(G) = S_k(\bar{G}) = \max\{t : \bar{d}_t \leq n - t + k\} = \max\{t : t - k - 1 \leq d_{n-t+1}\}$  and we are done.  $\square$

Note from previous theorem that if  $k \geq \Delta$ , then  $S_k(G) = n$  and  $\varphi_k(G) = 1$ . Also, if  $k \geq n - \delta - 1$ , then  $L_k(G) = n$  and  $\Omega_k(G) = 1$ . In this sense, the restrictions  $k \leq \Delta$  or  $k \geq n - \delta - 1$  needed in some of our theorems or observations are natural.

From Theorem 2.2, the following observation follows straightforward.

**Observation 2.3.** *Let  $G$  be a graph with minimum degree  $\delta$  and maximum degree  $\Delta$ . Then*

- (i)  $n - \Delta + k \leq S_k(G) \leq n - \delta + k$  for  $k \leq \Delta$ ;
- (ii)  $\delta + k + 1 \leq L_k(G) \leq \Delta + k + 1$  for  $k \leq n - \delta - 1$ ;
- (iii) if  $G$  is  $r$ -regular, then  $S_k(G) = n - r + k$  when  $k \leq r$  and  $L_k(G) = r + k + 1$  when  $k \leq n - r - 1$ .

*Proof.* (i) Let  $\delta = d_1 \leq d_2 \leq \dots \leq d_n = \Delta$  be the degree sequence of  $G$ . For  $k \leq \Delta$ ,  $d_{n-\Delta+k} \leq n - (n - \Delta + k) + k = \Delta$ . Therefore,  $n - \Delta + k \in \{s : d_s \leq n - s + k\}$  and thus  $n - \Delta + k \leq S_k(G)$ . Moreover, according to Theorem 2.2(iii), from  $S_k \in \{s : d_s \leq n - s + k\}$  it follows that  $\delta \leq d_{S_k(G)} \leq n - S_k(G) + k$ , that is,  $S_k(G) \leq n - \delta + k$ .

(ii) This follows from (i) applied to the graph  $\bar{G}$ .

(iii) This follows from (i) and (ii).  $\square$

Next we show a connection between  $L_0(G)$  and the chromatic number  $\chi(G)$  strengthening  $L_0(G) \geq \omega(G)$ . The analogon follows for  $S_0(G)$  and the clique-partition number  $\theta(G)$ .

**Observation 2.4.** *Let  $G$  be a graph. Then*

- (i)  $L_0(G) \geq \chi(G) \geq \omega(G)$ ;
- (ii)  $S_0(G) \geq \theta(G) \geq \alpha(G)$ .

*Proof.* (i) By a result of Powell and Welsh ([14], see also [11], p. 148),  $\chi(G) \leq \max\{\min\{i, d_i + 1\} : 1 \leq i \leq n\}$ , where  $d_1 \geq d_2 \geq \dots \geq d_n$  is the degree sequence of  $G$ . This can be rewritten with the conventional order  $d_1 \leq d_2 \leq \dots \leq d_n$  as  $\chi(G) \leq \max\{t : t \leq d_{n-t+1} + 1\}$ . Since, by the above theorem, the last expression is equal to  $L_0(G)$ , we obtain, together with Theorem 2.1, the desired inequality chain.

Another proof of  $L_0(G) \geq \chi(G)$  can be given the following way. Let  $V_1 \cup V_2 \cup \dots \cup V_r$  be an

$r$ -chromatic partition of  $V(G)$ , where  $r = \chi(G)$ . Suppose there is an index  $i$  such that every vertex  $v \in V_i$  has no neighbor in some set  $V_j$ , for an index  $j \neq i$ . Then we can distribute the vertices of  $V_i$  among the other sets  $V_j$ , obtaining thus an  $(r - 1)$ -chromatic coloring of  $G$ , which is a contradiction. Hence, for every  $1 \leq i \leq r$ , there is a vertex  $v_i \in V_i$  such that  $v_i$  has a neighbor in  $V_j$  for every  $1 \leq j \leq r$  and  $i \neq j$ . Therefore,  $\deg(v_i) \geq r - 1$  for  $1 \leq i \leq r$  and hence  $\{v_1, v_2, \dots, v_r\}$  is a large set, yielding  $\chi(G) = r \leq L_0(G)$ .  
(ii) This follows from (i) and  $S_0(G) = L_0(\overline{G})$ ,  $\theta(G) = \chi(\overline{G})$  and  $\alpha(G) = \omega(\overline{G})$ .  $\square$

We close this section with three observations about partitions of the vertex set of a graph into a  $k$ -small set and a  $k$ -large set.

**Observation 2.5.** *Let  $G$  be a graph. Then  $V(G)$  can be partitioned into a  $k$ -small set  $V_S$  and a  $k$ -large set  $V_L$ .*

*Proof.* Let  $d_1 \leq d_2 \leq \dots \leq d_n$  be the degree sequence of  $G$  and let  $j = S_k(G)$  be the largest index such that  $d_j \leq n - j + k$ . Let  $v_1, v_2, \dots, v_n$  be the vertices of  $G$  ordered according to its degree sequence. Set  $V_S = \{v_1, \dots, v_j\}$  and set  $V_L = V \setminus V_S$ . Clearly,  $|V_S| = j$  and  $|V_L| = n - j$ . By Theorem 2.2(iii),  $V_S$  is a maximum  $k$ -small set. Since  $j$  is the maximum index for which  $d_j \leq n - j + k$ , it follows that  $d_{j+1} > n - (j+1) + k$  and thus  $d_{j+1} \geq n - j + k$ . But then, for  $i \geq j + 1$ ,  $d_i \geq d_{j+1} \geq n - j + k = |V_L| + k > |V_L| - k - 1$ , and hence  $V_L$  is a  $k$ -large set. Note that already a partition into small and large sets suffices to prove the statement since any small set is a  $k$ -small set for  $k > 0$  and any large set is a  $k$ -large set for  $k > 0$ .  $\square$

From Observation 2.5 follows, in particular, that in every  $n$ -vertex graph there is either a  $k$ -small set on at least  $n/2$  vertices or a  $k$ -large set on at least  $n/2$  vertices.

**Observation 2.6.**  $n \leq L_k(G) + S_k(G) \leq n + 1 + 2k$  and this is sharp.

*Proof.* From Observation 2.5, we obtain directly the lower bound  $n \leq S_k(G) + L_k(G)$ . Let now  $A$  be a  $k$ -small set realizing  $S_k(G)$  and  $B$  a  $k$ -large set realizing  $L_k(G)$ . If  $A \cap B = \emptyset$ , then clearly  $|A| + |B| \leq n$ . Otherwise suppose there is a vertex  $u \in A \cap B$ . Then  $\deg(u) \leq n - |A| + k$  and  $\deg(u) \geq |B| - k - 1$ . Hence  $|B| - k - 1 \leq n - |A| + k$  and  $|A| + |B| \leq n + 1 + 2k$ .

To see the sharpness of the lower bound, let  $G_1$  be a graph on  $n_1 = 2q > 2(2k + 2)$  vertices whose vertex set can be split into an independent set  $V_S$  and a clique  $V_L$  with  $|V_S| = |V_L| = q$ , and such that their vertices are joined by  $k + 1$  pairwise disjoint perfect matchings. Then, the vertices in  $V_S$  have all degree  $k + 1$  and the vertices in  $V_L$  have all degree  $q + k$ . Hence, for the degree sequence  $d_1 \leq d_2 \leq \dots \leq d_{2q}$  of  $G_1$  we have  $d_q = k + 1 \leq n_1 - q + k = q + k$  and  $q + k = d_{q+1} > n_1 - (q + 1) + k = q + k - 1$ , from which follows that  $V_S$  is a maximum  $k$ -small set, by Theorem 2.2. Also from  $d_{q+1} \geq n_1 - q - k - 1 = q - k - 1$  and  $d_q = k + 1 < n_1 - q - k - 1 = q - k - 1$ , as  $q > 2k + 2$ , it follows by the same theorem that  $V_L$  is a maximum  $k$ -large set of  $G_1$ . Hence for this graph,  $n_1 = S_k + L_k$  holds. Finally, for

the sharpness of the upper bound, let  $G_2$  be a graph in which the largest  $2k + 1$  degrees in the degree sequence are  $k$ . An easy check reveals the required equality.  $\square$

**Observation 2.7.** *Let  $G$  be a graph on  $n$  vertices and  $e(G)$  edges. Then there is partition of  $V(G)$  into a  $k$ -small set  $V_S$  and a  $k$ -large set  $V_L$  such that  $|V_L| \leq \frac{1}{2}(k + 1 + \sqrt{(k + 1)^2 + 8e(G)})$  and hence  $|V_S| \geq n - \frac{1}{2}(k + 1 + \sqrt{(k + 1)^2 + 8e(G)})$ .*

*Proof.* Let  $V(G) = V_S \cup V_L$  be a partition into a  $k$ -small and a  $k$ -large set and let  $p = |V_L|$ . Then  $2e(G) \geq \sum_{v \in V_L} \deg(v) \geq p(p - k - 1)$ . Solving the quadratic inequality, we obtain  $p \leq \frac{1}{2}(k + 1 + \sqrt{(k + 1)^2 + 8e(G)})$ .  $\square$

### 3 Algorithms for $\varphi_k(G)$ and $\Omega_k(G)$

In this section, we will present two algorithms with which we will be able to calculate  $\varphi_k(G)$  and  $\Omega_k(G)$  for a graph  $G$ . For this, we consider any sequence of  $m$  integers  $A = \{0 \leq a_1 \leq \dots \leq a_m \leq n - 1\}$  (not necessarily graphic). Now we want to break the sequence into  $k$ -small subsequences. With this aim, we apply the following algorithm.

#### Algorithm 1

INPUT:  $A$

STEP 1: Set  $i := 0$ ,  $R_0 := A$ .

STEP 2: Repeat

- (1)  $n_i := |R_i|$
- (2)  $p_i := \min\{n_i, n - a_{n_i} + k\}$
- (3)  $A_{i+1} := \{a_{n_i - p_i + 1}, a_{n_i - p_i + 2}, \dots, a_{n_i}\}$
- (4)  $R_{i+1} := R_i \setminus A_{i+1}$
- (5)  $i := i + 1$

until  $R_i = \emptyset$ .

OUTPUT:  $s := i$ ,  $A_1, A_2, \dots, A_s$ .

Here,  $i$  stands for the current step number;  $R_i$  is the set of remaining elements and  $n_i$  its cardinality;  $A_{i+1}$  is the new subsequence constructed in step  $i$ ; and, on the output,  $s$  it is the number of constructed subsequences  $A_i$ .

**Theorem 3.1.** *Let  $A = \{0 \leq a_1 \leq \dots \leq a_m \leq n - 1\}$  be a sequence of  $m$  integers. Then Algorithm 1 under input  $A$  yields a minimum partition of  $A$  into  $s$   $k$ -small subsequences  $A_1, A_2, \dots, A_s$ .*

*Proof.* Clearly,  $A_i$  is a subsequence of  $A$  for  $i = 1, 2, \dots, s$ . By construction, in each step  $i$ ,  $A_{i+1} \subseteq R_i = R_{i-1} \setminus A_i$  and so the  $A_i$ 's are pairwise disjoint. Moreover, the last step  $s$  is attained when  $R_s = \emptyset$ , i.e.,  $A_s = R_{s-1}$ , meaning that  $A_s$  consists of all remaining elements of  $A$ . Hence,  $A_1, A_2, \dots, A_s$  is a splitting of  $A$  into subsequences. We now proceed to prove that, in each step  $i \geq 0$ , the produced subsequence  $A_{i+1}$  is  $k$ -small. We distinguish between the two possible situations:

(a)  $p_i = n_i \leq n - a_{n_i} + k$ : Then,  $A_{i+1} = \{a_1, a_2, \dots, a_{n_i}\} = R_i$  and, for every  $a \in A_{i+1}$ , we have  $a \leq a_{n_i} = n - (n - a_{n_i} + k) + k \leq n - n_i + k = n - |A_{i+1}| + k$ . Thus,  $A_{i+1}$  is a  $k$ -small subsequence.

(b)  $p_i = n - a_{n_i} + k \leq n_i$ : Then,  $A_{i+1} := \{a_{n_i - (n - a_{n_i} + k) + 1}, a_{n_i - (n - a_{n_i} + k) + 2}, \dots, a_{n_i}\}$  and, for every  $a \in A_{i+1}$ , we have  $a \leq a_{n_i} = n - (n - a_{n_i} + k) + k = n - |A_{i+1}| + k$ . Thus,  $A_{i+1}$  is a  $k$ -small subsequence of  $A$ .

Finally we shall prove that the output  $s$  given by Algorithm 1 is the minimum number of  $k$ -small subsequences in which  $A$  can be partitioned. Let  $A'_1, A'_2, \dots, A'_q$  be an optimal splitting of  $A$  into  $k$ -small sequences, i.e. such that  $q$  is minimum. Then clearly  $q \leq s$ . Let  $C_i = \max\{a : a \in A'_i\}$  and, without loss of generality, assume that  $C_1 \geq C_2 \geq \dots \geq C_q$ . We will show by induction on  $i$  that  $a_{n_i} \leq C_i$ . Since clearly  $a_{n_1} = a_m = C_1$ , the base case is done. Assume that  $a_{n_i} \leq C_i$  for  $i = 1, 2, \dots, r$  and an  $r < q$ . Then, as  $A'_i$  is a  $k$ -small set, we have

$$n - |A_i| + k = a_{n_i} \leq C_i \leq n - |A'_i| + k,$$

implying that  $|A'_i| \leq |A_i|$ , for  $i = 1, 2, \dots, r$ . Suppose to the contrary that  $a_{n_{r+1}} > C_{r+1}$ . Then  $a_{n_{r+1}} \in \cup_{i=1}^r A'_i$ . As  $\sum_{i=1}^r |A'_i| \leq \sum_{i=1}^r |A_i|$  and, moreover,  $a_{n_{r+1}} \notin \cup_{i=1}^r A_i$  by construction, there has to be an element  $y$  which is contained in  $\cup_{i=1}^r A_i$  but not in  $\cup_{i=1}^r A'_i$ . Hence,  $y \in A'_j$  for some  $j \geq r+1$  and  $y \geq a_{n_{r+1}}$ . As  $C_j$  is the largest element in  $A'_j$ , we conclude that  $C_{r+1} \geq C_j \geq y \geq a_{n_{r+1}}$ , contradicting the assumption. Hence  $a_{n_{r+1}} \leq C_{r+1}$  and by induction it follows that  $a_{n_i} \leq C_i$  for all  $i = 1, 2, \dots, q$ . As above, this implies that  $|A'_i| \leq |A_i|$  for all  $i = 1, 2, \dots, q$ . Hence,

$$m = \sum_{i=1}^q |A'_i| \leq \sum_{i=1}^q |A_i| \leq \sum_{i=1}^s |A_i| = m,$$

from which we obtain  $q = s$ . Therefore, Algorithm 1 yields a partition of  $A$  into the minimum possible number of  $k$ -small subsequences  $A_1, A_2, \dots, A_s$ .  $\square$

**Observation 3.2.** *Algorithm 1 can be written recursively by defining a function  $f$  which will give the partition of an arbitrary sequence into  $k$ -small subsequences:*

STEP 1: Set  $f(\emptyset) = \emptyset$ .

STEP 2:

$$f(\{0 \leq a_1 \leq \dots \leq a_m \leq n-1\}) = \{ \{a_{m - \min\{m, n - a_m + k\} + 1}, \dots, a_m\} \} \\ \cup f(\{0 \leq a_1 \leq \dots \leq a_{\min\{m, n - a_m + k\}} \leq n-1\})$$

When  $m = n$  and  $d_1 \leq d_2 \leq \dots \leq d_n$  is the degree sequence of a graph  $G$ , we can use Algorithm 1 to find a partition of  $V(G)$  into the minimum possible number of  $k$ -small sets.

**Corollary 3.3.** *Let  $G$  be a graph and  $d_1 \leq \dots \leq d_n$  its degree sequence. Let  $A = \{0 \leq d_1 \leq \dots \leq d_n \leq n-1\}$  and let  $V_1, V_2, \dots, V_s$  be the sets of vertices corresponding to the degree*



subsequences  $A_1, A_2, \dots, A_s$  given by Algorithm 1 under input  $A$ . Then  $V_1 \cup V_2 \cup \dots \cup V_s$  is a partition of  $V(G)$  into  $s = \varphi_k(G)$   $k$ -small sets.

By the duality between  $k$ -small and  $k$ -large sequences and since  $\Omega_k(G) = \varphi_k(\overline{G})$ , we can modify Algorithm 1 to an algorithm that leads us to find the exact value of  $\Omega_k(G)$ . Again, consider any sequence of  $m$  integers  $B = \{0 \leq b_m \leq b_{m-1} \leq \dots \leq b_1 \leq n-1\}$  (not necessarily graphic).

## Algorithm 2

INPUT:  $B$

STEP 1: Set  $i := 0$ ,  $S_0 := B$ .

STEP 2: Repeat

- (1)  $n_i := |S_i|$
- (2)  $q_i := \min\{n_i, b_{n_i} + k + 1\}$
- (3)  $B_{i+1} := \{b_{n_i}, b_{n_i-1}, \dots, b_{n_i-q_i+1}\}$
- (4)  $S_{i+1} := S_i \setminus B_{i+1}$
- (5)  $i := i + 1$

until  $S_i = \emptyset$ .

OUTPUT:  $t := i$ ,  $B_1, B_2, \dots, B_t$ .

**Theorem 3.4.** Let  $B = \{0 \leq b_m \leq b_{m-1} \leq \dots \leq b_1 \leq n-1\}$  be a sequence of  $m$  integers. Then Algorithm 2 under input  $B$  yields a minimum partition of  $B$  into  $s$   $k$ -large subsequences  $B_1, B_2, \dots, B_t$ .

*Proof.* Let  $A = \overline{B} = \{0 \leq a_1 \leq a_2 \leq \dots \leq a_m \leq n-1\}$  be the complementary sequence to  $B$ , where  $a_i = n - b_i - 1$ . Then, from the application of Algorithm 1 under input  $A$  and of Algorithm 2 under input  $B$ , it follows:

- (i)  $R_0 = A = \overline{B} = \overline{S_0}$
- (ii)  $R_i = \overline{S_i}$ ,  $|R_i| = n_i = |S_i|$
- (iii)  $q_i = \min\{n_i, b_{n_i} + k + 1\} = \min\{n_i, n - (n-1 - b_{n_i}) + k\} = \min\{n_i, n - a_{n_i} + k\} = p_i$
- (iv)  $B_{i+1} = \{b_{n_i}, b_{n_i-1}, \dots, b_{n_i-q_i+1}\} = \{n - a_{n_i} - 1, n - \underline{a_{n_i-1}} - 1, \dots, n - a_{n_i-q_i+1} - 1\} = \{n - a_{n_i-q_i+1} - 1, \dots, n - a_{n_i-1} - 1, n - a_{n_i} - 1\} = \overline{A_{i+1}}$  and
- (v)  $S_{i+1} = S_i \setminus B_{i+1} = \overline{R_i} \setminus A_{i+1}$ .

Moreover,  $S_i = \emptyset$  if and only if  $R_i = \emptyset$  and thus the number of steps performed by Algorithm 1 under input  $A$  is the same as the number of steps performed by Algorithm 2 under input

$B$  and hence  $s = t$ . Since Algorithm 1 yields a partition of  $A = \overline{B}$  into the  $k$ -small sets  $A_1, A_2, \dots, A_s$ , the output  $B_1, B_2, \dots, B_t$  of Algorithm 2 is a partition of  $B$  into  $k$ -large sets.  $\square$

Again, when  $m = n$  and  $d_n \leq d_{n-1} \leq \dots \leq d_1$  is the degree sequence of a graph  $G$ , we can use Algorithm 2 to find a partition of  $V(G)$  into the minimum possible number of  $k$ -large sets.

**Corollary 3.5.** *Let  $G$  be a graph and  $d_n \leq \dots \leq d_1$  its degree sequence. Let  $B = \{0 \leq d_n \leq \dots \leq d_1 \leq n-1\}$  and let  $V_1, V_2, \dots, V_t$  be the sets of vertices corresponding to the subsequences  $B_1, B_2, \dots, B_t$  given by Algorithm 2 under input  $B$ . Then  $V_1 \cup V_2 \cup \dots \cup V_t$  is a partition of  $V(G)$  into  $t = \Omega_k(G)$   $k$ -large sets.*

## 4 Bounds on $\varphi_k(G)$ and $\Omega_k(G)$

**Theorem 4.1.** *Let  $G$  be a graph on  $n$  vertices and with average degree  $d$ . Then*

$$(i) \quad \varphi_k(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} \geq \frac{n}{n - d + k};$$

$$(ii) \quad \Omega_k(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + k + 1} \geq \frac{n}{d + k + 1}.$$

*Proof.* (i) Let  $V_1, V_2, \dots, V_t$  be a partition of  $V(G)$  into  $t = \varphi_k(G)$   $k$ -small sets and set  $|V_i| = n_i$ , for  $1 \leq i \leq t$ . Then, as  $\deg(v) \leq n - n_i + k$  for each  $v \in V_i$ , we have

$$\sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} = \sum_{i=1}^t \sum_{v \in V_i} \frac{1}{n - \deg(v) + k} \leq \sum_{i=1}^t \sum_{v \in V_i} \frac{1}{n_i} = t = \varphi_k(G)$$

Now, Jensen's inequality yields

$$\varphi_k(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v) + k} \geq \frac{n}{n - d + k}.$$

(ii) Since  $\Omega_k(G) = \varphi_k(\overline{G})$ , we obtain from (i)

$$\Omega_k(G) = \varphi_k(\overline{G}) \geq \sum_{v \in V(G)} \frac{1}{n - \deg_{\overline{G}}(v) + k} \geq \frac{1}{n - d(\overline{G}) + k},$$

which is equivalent to

$$\Omega_k(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + k + 1} \geq \frac{n}{d + k + 1}.$$

$\square$

Theorems 2.1 and 4.1 for  $k = 0$  imply the following corollary.

**Corollary 4.2.** *Let  $G$  be a graph on  $n$  vertices and average degree  $d$ . Then*

$$(i) \quad \omega(G) \geq \varphi(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v)} \geq \frac{n}{n - d};$$

$$(ii) \quad \alpha(G) \geq \Omega(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v) + 1} \geq \frac{n}{d + 1}.$$

The first explicit proof of  $\alpha(G) \geq \frac{n}{d+1}$  can be found in [7]. Note also that item (ii) of the previous corollary improves the Caro-Wei bound  $\alpha(G) \geq \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$  [2, 17]. Moreover, the bound  $\varphi(G) \geq \sum_{v \in V(G)} \frac{1}{n - \deg(v)}$  was given in [13]. From the result that  $\alpha(G) \geq \Omega(G)$ , one may ask if  $\alpha_k(G) \geq (k+1)\Omega_k(G)$  holds in general. However, this is in general wrong, as can be seen by the following counter example. Let  $n = (k+2)q$  for an integer  $q < k+1$  and let  $G = K_{1,n}$  be a star with  $n$  leaves. Then, clearly,  $\alpha_k(G) = n = (k+2)q$ . Moreover,  $\Omega_k(G) = \left\lceil \frac{n+1}{k+2} \right\rceil = q+1$ , since every  $k$ -large set containing a vertex of degree one has cardinality at most  $k+2$ . Hence, in this case we have  $\alpha_k(G) = (k+2)q < (k+1)(q+1) = (k+1)\Omega_k(G)$  for  $q < k+1$ .

In view of the above counter example the following problem seems natural.

**Problem.** Let  $G$  be a graph on  $n$  vertices. Is it true that

$$\alpha_k(G) \geq \sum_{v \in V(G)} \frac{k+1}{\deg(v) + k + 1} \geq \frac{n}{d(G) + k + 1}?$$

**Corollary 4.3.** *Let  $G$  be a graph on  $n$  vertices and  $e(G)$  edges. Then*

$$(i) \quad e(G) \leq \frac{1}{2} \left( n^2 - \frac{n^2}{\varphi_k(G)} + nk \right);$$

$$(ii) \quad e(G) \geq \frac{1}{2} \left( \frac{n^2}{\Omega_k(G)} - n(k+1) \right).$$

*Proof.* (i) From Theorem 4.1 (i) and the fact that  $nd = 2e(G)$ , it follows  $\varphi_k(G) \geq \frac{n}{n-d+k} = \frac{n^2}{n^2 - 2e(G) + kn}$ . Solving this inequality for  $e(G)$ , we obtain the desired result.

(ii) Similar as in (i), from Theorem 4.1 (ii) and the fact that  $nd = 2e(G)$ , it follows that  $\Omega_k(G) \geq \frac{n^2}{2e(G) + k(n+1)}$ . Solving the obtained inequality for  $e(G)$ , the result follows.  $\square$

In the special case  $k = 0$ , Corollary 4.3 yields  $e(G) \leq \frac{n^2(\varphi(G)-1)}{2\varphi(G)}$ . This bound is better than the bound  $e(G) \leq \frac{n^2(\omega(G)-1)}{2\omega(G)}$  from classical Turán's Theorem, because  $\omega(G) \geq \varphi(G)$ . To illustrate this by an example, let  $G$  be the graph obtained from the graph  $2K_n$  by adding  $n$  new independent edges between the two copies of  $K_n$ . Then  $\varphi(G) = 2$  and  $\omega(G) = n$ . From Turán's Theorem we have  $e(G) \leq 2n(n-1)$  and from Corollary 4.3(ii) follows that  $e(G) \leq n^2$ . The last inequality gives us the exact value of  $e(G)$ .

**Theorem 4.4.** *Let  $G$  be a graph on  $n$  vertices with minimum degree  $\delta$ , maximum degree  $\Delta$  and average degree  $d$ . Then:*

- (i)  $\left\lceil \frac{n}{n-d+k} \right\rceil \leq \varphi_k(G) \leq \left\lceil \frac{n}{n+k-\Delta} \right\rceil$ ;
- (ii)  $\left\lceil \frac{n}{d+k+1} \right\rceil \leq \Omega_k(G) \leq \left\lceil \frac{n}{\delta+k+1} \right\rceil$ ;
- (iii) If  $\frac{r-2}{r-1}n + k < d \leq \Delta \leq \frac{r-1}{r}n + k$ , then  $\varphi_k(G) = r$ ;
- (iv) If  $\frac{n}{r} - k - 1 \leq \delta \leq d < \frac{n}{r-1} - k - 1$ , then  $\Omega_k(G) = r$ ;
- (v) If  $G$  is  $r$ -regular, then  $\varphi_k(G) = \left\lceil \frac{n}{n+k-r} \right\rceil$  and  $\Omega_k(G) = \left\lceil \frac{n}{r+k+1} \right\rceil$ .

*Proof.* (i) From Theorem 4.1(i), it follows directly

$$\varphi_k(G) \geq \left\lceil \frac{n}{n-d+k} \right\rceil.$$

Let now  $G$  be a graph on  $n$  vertices and with maximum degree  $\Delta$ . If  $k > \Delta$ , then  $\varphi_k(G) = 1$  and the right inequality side is obvious. So let  $k \leq \Delta$  and let  $A \subseteq V(G)$  be a set of cardinality  $n - \Delta + k$ . Then, for any  $v \in A$ ,  $\deg(v) \leq \Delta = n - (n - \Delta + k) + k = n - |A| + k$  and hence  $A$  is a  $k$ -small set. Now we will partition  $V(G) \setminus A$  into  $k$ -small sets. Note that  $|V(G) \setminus A| = \Delta - k$ . So take a partition  $V_1, V_2, \dots, V_t$  of  $V(G) \setminus A$  into  $t = \left\lceil \frac{\Delta-k}{n-\Delta+k} \right\rceil$  sets such that  $|V_i| = n - \Delta + k$  for  $i = 1, 2, \dots, t-1$  and  $|V_t| \leq n - \Delta + k$ . Since, for every vertex  $v \in V_i$ ,  $\deg(v) \leq \Delta = n - (n - \Delta + k) + k \leq n - |V_i| + k$ ,  $V_i$  is a  $k$ -small set, for  $1 \leq i \leq t$ . Hence  $A \cup V_1 \cup V_2 \cup \dots \cup V_t$  is a partition of  $V(G)$  into  $1 + t = 1 + \left\lceil \frac{\Delta-k}{n-\Delta+k} \right\rceil = \left\lceil \frac{n}{n-\Delta+k} \right\rceil$   $k$ -small sets, and thus

$$\varphi_k(G) \leq \left\lceil \frac{n}{n-\Delta+k} \right\rceil.$$

(ii) Theorem 4.1(ii) yields

$$\Omega_k(G) \geq \left\lceil \frac{n}{d+k+1} \right\rceil.$$

The other inequality side is obtained from (i) through  $\Omega_k(G) = \varphi_k(\overline{G}) \leq \left\lceil \frac{n}{n+k-\Delta(\overline{G})} \right\rceil = \left\lceil \frac{n}{\delta+k+1} \right\rceil$ .

(iii) If  $\frac{r-2}{r-1}n + k < d \leq \Delta \leq \frac{r-1}{r}n + k$ , we obtain from (i)

$$r-1 = \left\lceil \frac{n}{n-\frac{r-2}{r-1}n} \right\rceil < \left\lceil \frac{n}{n+k-d} \right\rceil \leq \varphi_k(G) \leq \left\lceil \frac{n}{n+k-\Delta} \right\rceil \leq \left\lceil \frac{n}{n-\frac{r-1}{r}n} \right\rceil = r$$

and thus  $\varphi_k(G) = r$ .

(iv) If  $\frac{n}{r} - k - 1 \leq \delta \leq d < \frac{n}{r-1} - k - 1$ , we obtain from (ii)

$$r-1 = \left\lceil \frac{n}{\frac{n}{r-1}} \right\rceil < \left\lceil \frac{n}{d+k+1} \right\rceil \leq \Omega_k(G) \leq \left\lceil \frac{n}{\delta+k+1} \right\rceil \leq \left\lceil \frac{n}{\frac{n}{r}} \right\rceil = r$$

and thus  $\Omega_k(G) = r$ .

(v) Recall from (i) that  $\left\lceil \frac{n}{n+k-d} \right\rceil \leq \varphi_k(G) \leq \left\lceil \frac{n}{n+k-\Delta} \right\rceil$  and thus, if  $d = \Delta = r$ , we have  $\varphi_k(G) = \left\lceil \frac{n}{n+k-r} \right\rceil$ . Analogously, item (ii) yields  $\Omega_k(G) = \left\lceil \frac{n}{r+k+1} \right\rceil$ .  $\square$

## 5 More applications to $\alpha(G)$ and $\omega(G)$

**Theorem 5.1.** *Let  $G$  be a graph on  $n$  vertices and with minimum degree  $\delta$  and maximum degree  $\Delta$ . Then*

$$(i) \quad \alpha_k(G) \leq S_k(G) \leq \frac{n-\Delta+k}{2} + \sqrt{\frac{(n-\Delta+k)^2}{4} + n\Delta - 2e(G)};$$

$$(ii) \quad \omega_k(G) \leq L_k(G) \leq \frac{\delta+k+1}{2} + \sqrt{\frac{(\delta+k+1)^2}{4} - n\delta + 2e(G)}.$$

Moreover, all bounds are sharp for regular graphs.

*Proof.* (i) Let  $A$  be a maximum  $k$ -small set and let  $\Delta$  be the maximum degree of  $G$ . Then  $\deg(v) \leq n - |A| + k$  for all  $v \in A$ . Then

$$\begin{aligned} 2e(G) &= \sum_{v \in V(G)} \deg(v) = \sum_{v \in A} \deg(v) + \sum_{v \in V(G) \setminus A} \deg(v) \\ &\leq (n - |A| + k)|A| + \Delta(n - |A|) \\ &= -|A|^2 + (n - \Delta + k)|A| + n\Delta, \end{aligned}$$

which implies that  $|A|^2 - (n - \Delta + k)|A| - n\Delta + 2e(G) \leq 0$ . Solving the quadratic inequality, we obtain the desired bound

$$\alpha_k(G) \leq S_k(G) \leq \frac{n - \Delta + k}{2} + \sqrt{\frac{(n - \Delta + k)^2}{4} + n\Delta - 2e(G)}.$$

Finally, if  $G$  is  $r$ -regular, by Observation 2.3(iii), all inequalities become equalities.

(ii) This follows from  $\omega_k(G) = \alpha_k(\overline{G})$  and item (i).  $\square$

The following corollary is straightforward from previous theorem and Observation 2.4.

**Corollary 5.2.** *Let  $G$  be a graph on  $n$  vertices, with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then*

$$(i) \quad \alpha(G) \leq \theta(G) \leq S_0(G) \leq \left\lceil \frac{n-\Delta}{2} + \sqrt{\frac{(n-\Delta)^2}{4} + n\Delta - 2e(G)} \right\rceil \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor;$$

$$(ii) \quad \omega(G) \leq \chi(G) \leq L_0(G) \leq \left\lceil \frac{\delta+1}{2} + \sqrt{\frac{(\delta+1)^2}{4} - n\delta + 2e(G)} \right\rceil \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right\rfloor.$$

*Proof.* (i) This follows from Observation 2.4(i) and Theorem 5.1(i) setting  $k = 0$ . The last inequality follows because the expression is monotone increasing with  $\Delta$  and  $\Delta \leq n - 1$ .  
(ii) This follows from (i), Observation 2.4(ii) and  $L_0(G) = S_0(\overline{G})$ .  $\square$

Note that item (i) of Corollary 5.2 is a refinement of the Hansen-Zheng bound [10] which states that  $\alpha(G) \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor$ . The inequality  $\chi(G) \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right\rfloor$  also is well known (cf. Proposition 5.2.1 in [5]).

We will need the following notation. For a set  $A$  of vertices of a graph  $G$ , let  $d_r(A) = \sqrt[r]{\frac{1}{|A|} \sum_{v \in A} \deg^r(v)}$ . When  $r = 1$ , we will set  $d(A)$  for  $d_1(A)$  and when  $A = V(G)$ , we will set  $d_r(G)$  instead of  $d_r(V(G))$ . Note that  $d(G)$  is the average degree of  $G$ . In the following, we will show that the inequality  $\varphi(G) \geq \frac{n}{n-d(G)}$  given in Corollary 4.2 can be improved when  $d(G)$  is substituted by  $d_3(G)$ . However, we will also show that, for  $r \geq 4$ ,  $d(G)$  will not be able to be replaced by  $d_r(G)$  that easily. First, we need to prove the following lemma.

**Lemma 5.3.** *Let  $\beta_1, \beta_2, \dots, \beta_r \in [0, 1]$  be real numbers such that  $\beta_1 + \beta_2 + \dots + \beta_r \leq r - 1$ . Then*

$$\sum_{i=1}^r (1 - \beta_i) \beta_i^r \leq \left( \frac{r-1}{r} \right)^r \quad (1)$$

*and equality holds if and only if  $\beta_1 = \beta_2 = \dots = \beta_r = \frac{r-1}{r}$ .*

*Proof.* If  $r = 1$ , then  $\beta_1 = 0$  and the inequality is obvious. Let  $r \geq 2$ . We consider the function  $f(x) = (1 - x)x^{r-1}$ ,  $x \geq 0$ . From  $f'(x) = x^{r-2}((r-1) - rx)$  we see that  $f(x)$  attains its absolute maximum exactly when  $x = \frac{r-1}{r}$  and thus

$$f(x) \leq f\left(\frac{r-1}{r}\right) = \frac{1}{r} \left(\frac{r-1}{r}\right)^{r-1}.$$

Hence, we have

$$(1 - \beta_i) \beta_i^r = (1 - \beta_i) \beta_i^{r-1} \beta_i \leq \frac{1}{r} \left(\frac{r-1}{r}\right)^{r-1} \beta_i, \quad i = 1, 2, \dots, r.$$

Now the condition  $\beta_1 + \beta_2 + \dots + \beta_r \leq r - 1$  yields

$$\sum_{i=1}^r (1 - \beta_i) \beta_i^r \leq \frac{1}{r} \left(\frac{r-1}{r}\right)^{r-1} (\beta_1 + \beta_2 + \dots + \beta_r) \leq \left(\frac{r-1}{r}\right)^r$$

and the desired inequality holds. Suppose now that we have equality in (1). Then we have equality in all the above given inequalities and hence

$$(1 - \beta_i) \beta_i^{r-1} = \frac{1}{r} \left(\frac{r-1}{r}\right)^{r-1}, \quad i = 1, 2, \dots, r,$$

implying thus  $\beta_1 = \beta_2 = \dots = \beta_r = \frac{r-1}{r}$ .  $\square$

**Theorem 5.4.** *Let  $G$  be a graph on  $n$  vertices. Then, the following statements hold:*

- (i) *For every integer  $r \leq \varphi(G)$ ,  $\varphi(G) \geq \frac{n}{n-d_r(G)}$ . Moreover, equality holds if and only if  $G$  is an  $\frac{n(\varphi(G)-1)}{\varphi(G)}$ -regular graph.*
- (ii)  *$\varphi(G) \geq \frac{n}{n-d_3(G)}$ . Moreover, equality holds if and only if  $G$  is an  $\frac{n(\varphi(G)-1)}{\varphi(G)}$ -regular graph.*
- (iii) *If  $\varphi(G) \neq 2$ , then  $\varphi(G) \geq \frac{n}{n-d_4(G)}$ . Moreover, there exists a graph  $G$  for which  $\varphi(G) = 2$  and  $\varphi(G) < \frac{n}{n-d_4(G)}$ .*

*Proof.* (i) Since  $d_{r-1}(G) \leq d_r(G)$  for all  $r \leq \varphi(G)$ , it is enough to prove  $\varphi(G) \geq \frac{n}{n-d_{\varphi(G)}(G)}$ . Let  $\varphi(G) = \varphi$  and let  $V(G) = V_1 \cup V_2 \cup \dots \cup V_{\varphi}$  be a partition of  $V(G)$  into small sets and let  $n_i = |V_i|$ ,  $1 \leq i \leq \varphi$ . As  $\deg(v) \leq n - n_i$  for every  $v \in V_i$  and  $1 \leq i \leq \varphi$ , we have

$$(d_{\varphi}(G))^{\varphi} n = \sum_{v \in V(G)} \deg^{\varphi}(v) = \sum_{i=1}^{\varphi} \sum_{v \in V_i} \deg^{\varphi}(v) \leq \sum_{i=1}^{\varphi} n_i (n - n_i)^{\varphi}. \quad (2)$$

Setting  $\beta_i = 1 - \frac{n_i}{n}$  for  $1 \leq i \leq \varphi$ , the inequality above can be rewritten as

$$(d_{\varphi}(G))^{\varphi} n = \sum_{v \in V(G)} \deg^{\varphi}(v) \leq n^{\varphi+1} \sum_{i=1}^{\varphi} (1 - \beta_i) \beta_i^{\varphi}. \quad (3)$$

Since  $\beta_1 + \beta_2 + \dots + \beta_{\varphi} = \varphi - 1$ , Lemma 5.3 yields  $d_{\varphi}(G) \leq \frac{n(\varphi-1)}{\varphi}$ , from which follows the desired inequality  $\varphi(G) = \varphi \geq \frac{n}{n-d_{\varphi}(G)}$ . Hence we have proved

$$\varphi \geq \frac{n}{n-d_{\varphi}} \geq \frac{n}{n-d_{\varphi-1}(G)} \geq \dots \geq \frac{n}{n-d_r(G)}. \quad (4)$$

for any  $1 \leq r \leq \varphi(G)$ .

Suppose now that we have  $\varphi(G) = \frac{n}{n-d_r(G)}$  for some  $1 \leq r \leq \varphi = \varphi(G)$ . Then, we have equality all over the inequality chain (4). In particular,  $\varphi = \frac{n}{n-d_{\varphi}(G)}$ , which is equivalent to  $d_{\varphi} = \frac{n(\varphi-1)}{\varphi}$ , and hence we have equality in (2) and (3), too. From the equality in (2), it follows  $\deg(v) = n - n_i$  for  $v \in V_i$ ,  $1 \leq i \leq \varphi$ . From  $d_r = \frac{n(\varphi-1)}{\varphi}$  and the equality in (3), we see that in (1) there is equality, too. Moreover, from Lemma 5.3 it follows (with  $r = \varphi$ ) that  $\beta_i = \frac{\varphi-1}{\varphi}$  for  $1 \leq i \leq \varphi$  and thus  $n_i = \frac{n}{\varphi}$  and  $\varphi$  divides  $n$ . Hence,  $\deg(v) = n - n_i = \frac{n(\varphi-1)}{\varphi}$  for all  $v \in V_i$  and  $1 \leq i \leq \varphi$ , turning out that  $G$  is  $\frac{n(\varphi-1)}{\varphi}$ -regular. Conversely, if  $G$  is  $\frac{n(\varphi-1)}{\varphi}$ -regular, then evidently  $d_{\varphi}(G) = \frac{n(\varphi-1)}{\varphi} = d_r(G)$  for every  $r \leq \varphi$ . Then from Theorem 4.4

we have  $\varphi(G) = \left\lceil \frac{n}{n-\frac{n(\varphi-1)}{\varphi}} \right\rceil = \frac{n}{n-\frac{n(\varphi-1)}{\varphi}} = \frac{n}{n-d_r(G)}$ .

(ii) If  $\varphi = \varphi(G) \geq 3$ , then from item (i) we have  $\varphi(G) \geq \frac{n}{n-d_3(G)}$  with equality if and only if  $G$  is  $\frac{(\varphi-1)n}{\varphi}$ -regular. It remains to consider the cases  $\varphi(G) = 1$  and  $\varphi(G) = 2$ . Note that  $\varphi(G) = 1$  holds if and only if  $G = \overline{K_n}$ . Hence, in this case  $d_3(G) = 0$  and

$\varphi(G) = 1 = \frac{n}{n-d_3(G)}$ . So assume that  $\varphi(G) = 2$  and let  $V(G) = V_1 \cup V_2$  be a partition of  $V(G)$  into two small sets. Setting  $|V_1| = n_1$  and  $|V_2| = n_2 = n - n_1$ , we have

$$\sum_{v \in V(G)} \deg^3(v) = \sum_{v \in V_1} \deg^3(v) + \sum_{v \in V_2} \deg^3(v) \leq n_1(n - n_1)^3 + n_2(n - n_2)^3 = n_1 n_2 (n^2 - 2n_1 n_2).$$

The last expression takes its maximum when  $n_1 n_2 = \frac{n^2}{4}$ . Hence, it follows  $\sum_{v \in V(G)} \deg^3(v) \leq \frac{n^4}{8}$  and thus  $d_3(G) \leq \frac{n}{2}$ , which yields  $\frac{n}{n-d_3(G)} \leq 2 = \varphi(G)$ .

Now suppose that  $\varphi(G) = 2 = \frac{n}{n-d_3(G)}$ . Then we have equality in the inequality given above.

Hence,  $n_1 n_2 = \frac{n^2}{4}$  and  $\deg(v) = n - n_i$  for  $v \in V_i$ ,  $i = 1, 2$ . Therefore,  $n_1 = n_2 = \frac{n}{2} = \frac{n(\varphi-1)}{\varphi}$  and  $G$  is an  $\frac{n(\varphi-1)}{\varphi}$ -regular graph. On the other side, if  $G$  is an  $\frac{n}{2}$ -regular graph, then  $d_3(G) = \frac{n}{2}$  and, from Theorem 4.4 (v),  $\varphi(G) = 2$ . Hence  $\varphi(G) = 2 = \frac{n}{n-d_3(G)}$ .

(iii) The case  $\varphi(G) = 1$  is trivial. If  $\varphi(G) \geq 4$ , then the statement follows from item (i). The case  $\varphi(G) = 3$  can be proved by straightforward calculations using Lagrange multipliers. As in the case (i), a partition of  $V(G)$  into  $\varphi(G) = 3$  small sets  $V_1, V_2, V_3$  with  $|V_1| = n_1$ ,  $|V_2| = n_2$  and  $|V_3| = n_3$  leads to the inequality

$$\left( \frac{d_4(G)}{n} \right)^4 \leq \sum_{i=1}^3 (1 - \beta_i) \beta_i^4 = f(\beta_1, \beta_2, \beta_3),$$

where  $\beta_i = 1 - \frac{n_i}{n}$  and clearly  $\beta_1 + \beta_2 + \beta_3 = 2$  and  $\beta_i \in [0, 1]$ , for  $i = 1, 2, 3$ . We will show that  $f(\beta_1, \beta_2, \beta_3) \leq \left(\frac{2}{3}\right)^4$ . Let

$$F(\beta_1, \beta_2, \beta_3, \lambda) = \sum_{i=1}^3 (1 - \beta_i) \beta_i^4 + \lambda(\beta_1 + \beta_2 + \beta_3 - 2)$$

be the Lagrange function. The extremal points are either solutions of the system

$$\begin{cases} \frac{\partial F}{\partial \beta_i} = 4\beta_i^3 - 5\beta_i^4 - \lambda = 0, & i = 1, 2, 3 \\ \frac{\partial F}{\partial \lambda} = \beta_1 + \beta_2 + \beta_3 - 2 = 0 \end{cases}$$

or they are points on the border. We shall prove that the system has no solution in which  $\beta_1, \beta_2, \beta_3$  are pairwise distinct. Let us suppose the contrary. Then  $\beta_1, \beta_2, \beta_3$  are roots of  $g(x) = 5x^4 - 4x^3 + \lambda$ . As  $\beta_1 + \beta_2 + \beta_3 = 2$  from Vieta's formula follows that the fourth root of  $g$  is  $-\frac{6}{5}$ . Therefore  $\lambda = -12\left(\frac{6}{5}\right)^2$  and so  $g(x)$  has only two real roots, which is a contradiction. Let  $(\beta_1, \beta_2, \beta_3)$  be an extremal point which is not on the border. As  $\beta_1, \beta_2, \beta_3$  are solutions of the system, we can suppose that  $\beta_1 = 2\beta$  and  $\beta_2 = \beta_3 = 1 - \beta$ , where  $\beta \in [0, \frac{1}{2}]$ . Then

$$f(\beta_1, \beta_2, \beta_3) = f(\beta) = -30\beta^5 + 8\beta^4 + 12\beta^3 - 8\beta^2 + 2\beta$$

and

$$f'(\beta) = -2(3\beta - 1)(25\beta^3 + 3\beta^2 - 5\beta + 1).$$

$f'$  has two real roots,  $\frac{1}{3}$  and another one negative. Therefore,  $f$  attains its maximum  $\left(\frac{2}{3}\right)^4$  in  $[0, \frac{1}{2}]$  exactly when  $\beta = \frac{1}{3}$ . It is easy to see that the maximum on the border is  $\frac{1}{12}$ , which



is strictly smaller than  $(\frac{2}{3})^4$ . Hence, we have  $\left(\frac{d_4(G)}{n}\right)^4 \leq \left(\frac{2}{3}\right)^4 = \left(\frac{\varphi(G)-1}{\varphi(G)}\right)$ , implying thus that  $\varphi(G) \geq \frac{n}{n-d_4(G)}$ .

Consider now the graph  $G = K_{1,9}$ . It is clear that  $\varphi(G) = 2$ ,  $d_4(G) = \sqrt[4]{657} > 5$ . Therefore  $2 = \varphi(G) < \frac{10}{10-d_4(G)}$ .  $\square$

**Corollary 5.5.** *Let  $G$  be a graph on  $n$  vertices. Then, the following statements hold:*

- (i) *For every integer  $r \leq \varphi(G)$ ,  $\omega(G) \geq \frac{n}{n-d_r(G)}$  and equality holds if and only if  $G$  is a complete  $\omega(G)$ -partite Turán graph  $K_{\frac{n}{\omega(G)}, \frac{n}{\omega(G)}, \dots, \frac{n}{\omega(G)}}$ .*
- (ii)  *$\omega(G) \geq \frac{n}{n-d_3(G)}$  and equality holds if and only if  $G$  is a complete  $\omega(G)$ -partite Turán graph  $K_{\frac{n}{\omega(G)}, \frac{n}{\omega(G)}, \dots, \frac{n}{\omega(G)}}$ .*
- (iii) *If  $\varphi(G) \neq 2$ , then  $\omega(G) \geq \frac{n}{n-d_4(G)}$ .*

*Proof.* (i) From Theorems 2.1 and 5.4(i), we have  $\omega(G) \geq \varphi(G) \geq \frac{n}{n-d_r(G)}$ . Suppose now that  $\omega(G) = \frac{n}{n-d_r(G)}$ . Then we have equality in Theorem 5.4(i). Thus, setting  $\varphi(G) = \omega(G) = \omega$ ,  $G$  is  $\frac{n(\omega-1)}{\omega}$ -regular and  $e(G) = \frac{n^2(\omega-1)}{2\omega}$ . Since  $\omega(G) = \omega$ , from Turán's Theorem it follows that  $G$  is a complete  $\omega$ -chromatic regular graph, i.e.  $G$  is a complete  $\omega$ -partite Turán graph  $K_{\frac{n}{\omega}, \frac{n}{\omega}, \dots, \frac{n}{\omega}}$ . Conversely, if  $G$  is the complete  $\omega$ -partite Turán graph  $K_{\frac{n}{\omega}, \frac{n}{\omega}, \dots, \frac{n}{\omega}}$ , then evidently  $d_r(G) = \frac{n(\omega-1)}{\omega}$  and hence  $\omega(G) = \omega = \frac{n}{n-d_r(G)}$ .

(ii) From Theorems 2.1 and 5.4(ii), we have  $\omega(G) \geq \varphi(G) \geq \frac{n}{n-d_3(G)}$ . Suppose now that  $\omega = \omega(G) = \frac{n}{n-d_3(G)}$ . Then  $\varphi(G) = \frac{n}{n-d_3(G)}$ , i.e. we have equality in Theorem 5.4(ii). Thus, setting  $\varphi(G) = \omega(G) = \omega$ ,  $G$  is  $\frac{n(\omega-1)}{\omega}$ -regular and  $e(G) = \frac{n^2(\omega-1)}{2\omega}$ . Since  $\omega(G) = \omega$ , from Turán's Theorem it follows that  $G$  is a complete  $\omega$ -chromatic regular graph, i.e.  $G = K_{\frac{n}{\omega}, \frac{n}{\omega}, \dots, \frac{n}{\omega}}$ . Conversely, if  $G$  is the complete  $\omega$ -partite Turán graph  $K_{\frac{n}{\omega}, \frac{n}{\omega}, \dots, \frac{n}{\omega}}$ , then evidently  $d_3(G) = \frac{n(\omega-1)}{\omega}$  and hence  $\omega(G) = \omega = \frac{n}{n-d_3(G)}$ .

(iii) This follows from Theorems 2.1 and 5.4(iii).  $\square$

Note that Theorem 5.4(ii) improves the bound  $\varphi(G) \geq \frac{n}{n-d_2(G)}$  given in [1] and Corollary 5.5(ii) is better than the inequality  $\omega(G) \geq \frac{n}{n-d_2(G)}$ , given in [6] and later in [1] where the proof was corrected.

Since  $\alpha(G) = \omega(\overline{G})$  and  $\Omega(G) = \varphi(\overline{G})$ , we have the following corollaries.

**Corollary 5.6.** *Let  $G$  be a graph on  $n$  vertices. Then, the following statements hold:*

- (i) *For every integer  $r \leq \Omega(G)$ ,  $\Omega(G) \geq \frac{n}{n-d_r(\overline{G})}$ . Moreover, equality holds if and only if  $G$  is an  $(\frac{n}{\Omega(G)} - 1)$ -regular graph.*
- (ii)  *$\Omega(G) \geq \frac{n}{n-d_3(\overline{G})}$ . Moreover, equality holds if and only if  $G$  is an  $(\frac{n}{\Omega(G)} - 1)$ -regular graph.*

(iii) If  $\Omega(G) \neq 2$ , then  $\Omega(G) \geq \frac{n}{n-d_4(\overline{G})}$ . Moreover, there exists a graph  $G$  for which  $\varphi(G) = 2$  and  $\Omega(G) < \frac{n}{n-d_4(\overline{G})}$ .

**Corollary 5.7.** *Let  $G$  be a graph on  $n$  vertices. Then, the following statements hold:*

- (i) *For every integer  $r \leq \Omega(G)$ ,  $\alpha(G) \geq \frac{n}{n-d_r(\overline{G})}$  and equality holds if and only if  $G$  is the union of  $\alpha(G)$  copies of  $K_{\frac{n}{\alpha(G)}}$ .*
- (ii)  *$\alpha(G) \geq \frac{n}{n-d_3(\overline{G})}$  and equality holds if and only if  $\alpha(G)$  copies of  $K_{\frac{n}{\alpha(G)}}$ .*
- (iii) *If  $\Omega(G) \neq 2$ , then  $\alpha(G) \geq \frac{n}{n-d_4(\overline{G})}$ .*

## 6 Variations of small and large sets

Let  $G$  be a graph on  $n$  vertices and  $A$  a subset of  $V(G)$ . We call  $A$   $\alpha$ -small if  $\sum_{v \in A} \frac{1}{n-\deg(v)} \leq 1$  and  $\beta$ -small if  $d(A) \leq n - |A|$ . Now we observe the following.

**Observation 6.1.** *In a graph  $G$ , every small set is an  $\alpha$ -small set and every  $\alpha$ -small set is a  $\beta$ -small set.*

*Proof.* If  $A$  is a small set of  $G$ , then  $n - \deg(v) \geq |A|$  for every vertex  $v \in A$  and we have  $\sum_{v \in A} \frac{1}{n-\deg(v)} \leq \sum_{v \in A} \frac{1}{|A|} = 1$ . Hence,  $A$  is an  $\alpha$ -small set. Further, if  $A$  is an  $\alpha$ -small set of  $G$ , then  $1 \geq \sum_{v \in A} \frac{1}{n-\deg(v)} \geq \frac{|A|}{n-d(A)}$  by Jensen's inequality and hence  $d(A) \leq n - |A|$  and thus  $A$  is a  $\beta$ -small set.  $\square$

Let now  $\varphi^\alpha(G)$  and  $\varphi^\beta(G)$  be the minimum number of  $\alpha$ -small sets and, respectively,  $\beta$ -small sets in which  $V(G)$  can be partitioned. Further, let  $CW(G) = \sum_{v \in V(G)} \frac{1}{\deg(v)+1}$  be the Caro-Wei bound.

**Theorem 6.2.** *Let  $G$  be a graph on  $n$  vertices. Then*

- (i)  $\omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq \varphi^\beta(G) \geq \left\lceil \frac{n}{n-d(G)} \right\rceil$ ;
- (ii)  $\omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq CW(\overline{G}) \geq \left\lceil \frac{n}{n-d(G)} \right\rceil$ .

*Proof.* Since every small set is an  $\alpha$ -small set and every  $\alpha$ -small set is a  $\beta$ -small set and because of Theorem 2.1, we have the inequality chain  $\omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq \varphi^\beta(G)$ . Now we will prove the remaining bounds.

(i) Let  $t = \varphi^\beta(G)$  and let  $V(G) = A_1 \cup A_2 \cup \dots \cup A_t$  be a partition of  $V(G)$  into  $\beta$ -small sets. Then, using the definition of  $\beta$ -small set and Jensen's inequality, we obtain

$$nd(G) = 2e(G) = \sum_{v \in V(G)} \deg(v) = \sum_{i=1}^t \sum_{v \in A_i} \deg(v) \leq \sum_{i=1}^t (n - |A_i|)|A_i| \leq n \left( n - \frac{n}{t} \right).$$

Hence  $d(G) \leq n - \frac{n}{t} = n - \frac{n}{\varphi^\beta(G)}$ , which is equivalent to  $\varphi^\beta(G) \geq \frac{n}{n-d(G)}$ .

(ii) Let  $V(G) = A_1 \cup A_2 \cup \dots \cup A_t$  be a partition of  $V(G)$  into  $t = \varphi^\alpha(G)$   $\alpha$ -small sets. Then, Corollary 4.2(i) and the definition of  $\alpha$ -small set yield

$$\frac{n}{n-d(G)} \leq CW(\overline{G}) = \sum_{v \in V(G)} \frac{1}{n - \deg(v)} = \sum_{i=1}^t \sum_{v \in A_i} \frac{1}{n - \deg(v)} \leq t = \varphi^\alpha(G).$$

□

Let us consider an example. Let  $G$  be a graph obtained from  $2K_n$  by joining one of the vertices of the first copy of  $K_n$  to all the vertices of the second copy of  $K_n$ . Then  $\varphi(G) = 3$ ,  $CW(\overline{G}) = 3 - \frac{2}{n+1}$  and  $\varphi^\beta(G) = 2$ . In this case  $\varphi^\beta(G) \leq CW(\overline{G})$ . We do not know if  $\varphi^\beta(G) \leq CW(\overline{G})$  is always true.

The inequality chains given in Theorem 6.2(i) and (ii) together with the fact that  $2e(G) = nd(G)$  lead to the following corollary.

**Corollary 6.3.** *Let  $G$  be a graph on  $n$  vertices. Then*

$$(i) \quad e(G) \leq \frac{(\varphi^\beta(G)-1)n^2}{2\varphi^\beta(G)} \leq \frac{(\varphi^\alpha(G)-1)n^2}{2\varphi^\alpha(G)} \leq \frac{(\varphi(G)-1)n^2}{2\varphi(G)} \leq \frac{(\omega(G)-1)n^2}{2\omega(G)};$$

$$(ii) \quad e(G) \leq \frac{(CW(\overline{G})-1)n^2}{2CW(\overline{G})} \leq \frac{(\varphi^\alpha(G)-1)n^2}{2\varphi^\alpha(G)} \leq \frac{(\varphi(G)-1)n^2}{2\varphi(G)} \leq \frac{(\omega(G)-1)n^2}{2\omega(G)}.$$

As remarked for Corollary 4.3, the above bounds on  $e(G)$  are better than the bound  $e(G) \leq \frac{n^2(\omega(G)-1)}{2\omega(G)}$  from classical Turán's Theorem, because  $\omega(G) \geq \varphi(G) \geq \varphi^\alpha(G) \geq CW(\overline{G})$  and  $\varphi^\alpha(G) \geq \varphi^\beta(G)$ .

Analogous to  $\alpha$ -small and  $\beta$ -small sets, we can define  $\alpha$ -large and  $\beta$ -large sets. Let  $G$  be a graph on  $n$  vertices and  $B$  a subset of  $V(G)$ .  $B$  will be called  $\alpha$ -large if  $\sum_{v \in B} \frac{1}{\deg(v)+1} \leq 1$  and  $\beta$ -large if  $d(B) \geq |B| - 1$ . As for small sets, every large set is an  $\alpha$ -large set and every  $\alpha$ -large set is a  $\beta$ -large set. We also define  $\Omega^\alpha(G)$  and  $\Omega^\beta(G)$  as the minimum number of  $\alpha$ -large sets and, respectively,  $\beta$ -large sets in which  $V(G)$  can be partitioned.

Theorem 6.2 and Corollary 4.3 yield, together with the known facts that  $\alpha(G) = \omega(\overline{G})$ ,  $\Omega(G) = \varphi(\overline{G})$ ,  $\Omega^\alpha(G) = \varphi^\alpha(\overline{G})$  and  $\Omega^\beta(G) = \varphi^\beta(\overline{G})$ , the following corollaries.

**Corollary 6.4.** *Let  $G$  be a graph on  $n$  vertices. Then*

$$(i) \quad \alpha(G) \geq \Omega(G) \geq \Omega^\alpha(G) \geq \Omega^\beta(G) \geq \frac{n}{d(G)+1};$$

$$(ii) \quad \alpha(G) \geq \Omega(G) \geq \Omega^\alpha(G) \geq CW(G) \geq \frac{n}{d(G)+1};$$

**Corollary 6.5.** *Let  $G$  be a graph on  $n$  vertices. Then*

$$\begin{aligned}
(i) \quad e(G) &\geq \frac{n}{2} \left( \frac{n}{\Omega^\beta(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\Omega^\alpha(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\Omega(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\alpha(G)} - 1 \right); \\
(ii) \quad e(G) &\geq \frac{n}{2} \left( \frac{n}{\Omega^\beta(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\Omega^\alpha(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\Omega(G)} - 1 \right) \geq \frac{n}{2} \left( \frac{n}{\alpha(G)} - 1 \right).
\end{aligned}$$

Let  $S^\alpha(G)$  and  $S^\beta(G)$  be the maximum cardinality of an  $\alpha$ -small set and of a  $\beta$ -small set of  $G$ , respectively. Analogously, let  $L^\alpha(G)$  and  $L^\beta(G)$  be the maximum cardinality of an  $\alpha$ -large set and of a  $\beta$ -large set of  $G$ , respectively. We finish this section with the following theorem.

**Theorem 6.6.** *Let  $G$  be a graph on  $n$  vertices, with maximum degree  $\Delta$  and minimum degree  $\delta$ . Then*

$$\begin{aligned}
(i) \quad \alpha(G) &\leq S_0(G) \leq S^\alpha(G) \leq S^\beta(G) \\
&\leq \left\lfloor \frac{n-\Delta}{2} + \sqrt{\frac{(n-\Delta)^2}{4} + n\Delta - 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + n^2 - n - 2e(G)} \right\rfloor; \\
(ii) \quad \omega(G) &\leq L_0(G) \leq L^\alpha(G) \leq L^\beta(G) \\
&\leq \left\lfloor \frac{\delta+1}{2} + \sqrt{\frac{(\delta+1)^2}{4} - n\delta + 2e(G)} \right\rfloor \leq \left\lfloor \frac{1}{2} + \sqrt{\frac{1}{4} + 2e(G)} \right\rfloor.
\end{aligned}$$

*Proof.* The inequality chains  $\alpha(G) \leq S_0(G) \leq S^\alpha(G) \leq S^\beta(G)$  and  $\omega(G) \leq L_0(G) \leq L^\alpha(G) \leq L^\beta(G)$  follow from Theorem 2.2(i) for  $k = 0$  and Observation 6.1. The proof of the right side inequalities is analogous to the proof of the Theorem 5.1 in case  $k = 0$ .  $\square$

Note also that Corollary 5.2 follows from this theorem because of  $S_0(G) \leq S^\alpha(G)$  and  $L_0(G) \leq L^\alpha(G)$ .

## References

- [1] A. Bojilov, N. Nenov, An inequality for generalized chromatic graphs, *Mathematics and education in mathematics* (2012), 143–147, Proceedings of the Forty First Spring Conference of Union of Bulgarian Mathematics, Borovets, April 9–12, 2012.
- [2] Y. Caro, New results on the independence number, *Tech. Report, Tel-Aviv University* (1979).
- [3] Y. Caro, Z. Tuza, Improved lower bounds on  $k$ -independence. *J. Graph Theory* **15** (1991), 99–107.
- [4] M. Chellali, O. Favaron, A. Hansberg, L. Volkmann,  $k$ -domination and  $k$ -independence in graphs: A survey, *Graphs Combin.* **28** (2012), 1–55.
- [5] R. Diestel, Graph Theory, Series: Graduate Texts in Mathematics, Vol. 173, 4th ed. 2010. Corr. 3rd printing 2012, 2010, XVIII, 410 p. 123 illus.

- [6] C. Edwards and C. Elphick, Lower bounds for the clique and the chromatic number of a graph, *Discrete Appl. Math.* (1983) **5**, 51–64.
- [7] P. Erdős, T. Galai, On the minimal number of vertices representing the edges of a graph, *Publ. Math. Inst. Hung. Acad. Sci.* **6** (1961), 181–203.
- [8] O. Favaron,  $k$ -domination and  $k$ -independence in graphs. *Ars Combin.* **25** C (1988), 159–167.
- [9] O. Favaron, M. Mahéo, J.-F. Saclé, On the residue of a graph, *J. Graph Theory* **15** (1991), no. 1, 39–64.
- [10] P. Hansen, M. Zheng, Sharp bounds on the order, size and stability number of graphs, *Networks* **23** (1993), no. 2, 99–102.
- [11] F. Harary, *Graph Theory*, Addison-Wesley Publishing Co., Reading, Mass.-Menlo Park, Calif.-London 1969 ix+274 pp.
- [12] F. Jelen,  $k$ -Independence and the  $k$ -residue of a graph, *J. Graph Theory* **32** (1999), no. 3, 241–249.
- [13] N. Nenov, Improvement of Graph Theory Wei’s inequality, *Mathematics and education in mathematics* (2006), 191–194, Proceedings of the Thirty Fifth Spring Conference of Union of Bulgarian Mathematics, Borovets, April 5–8, 2006.
- [14] M. B. Powell, D. J. A. Welsh, An upper bound for the chromatic number of a graph and its application to timetabling problems *Comput. J.* **10** (1967), 85–87.
- [15] E. Triesch, Degree sequences of graphs and dominance order, *J. Graph Theory* **22** (1996), no. 1, 89–93.
- [16] P. Turán, Eine Extremalaufgabe aus der Graphentheorie, *Mat. Fiz. Lapok* (1941) **48**, 436–452.
- [17] V. K. Wei, A lower bound on the stability number of a simple graph, *Bell Laboratories Technical Memorandum*, 81–11217–9, Murray Hill, NJ (1981).
- [18] D. B. West, *Introduction to graph theory - Second edition*, Prentice Hall, Inc., Upper Saddle River, NJ, 2001, xx+588 pp.
- [19] W.W. Willis, Bounds for the independence number of a graph, MSc thesis, Virginia Commonwealth University, 2011.